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Bosonization of Three Dimensional Non-Abelian Fermion Field Theories

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Abstract

We discuss bosonization in three dimensions of an $SU(N)$ massive Thirring model in the low-energy regime. We find that the bosonized theory is related (but not equal) to $SU(N)$ Yang-Mills-Chern-Simons gauge theory. For free massive fermions bosonization leads, at low energies, to the pure $SU(N)$ (level $k = 1$) Chern-Simons theory.

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1 Introduction

In this paper we investigate the problem of identifying a bosonic equivalent of a theory of self-interacting fermions with $SU(N)$ symmetry in $2+1$ dimensions. In a previous publication [1], two of us showed that the low-energy sector of the $U(1)$ massive Thirring model in $2+1$ dimensions is equivalent to the Maxwell-Chern-Simons gauge theory. Here we extend that analysis to a theory with non-abelian $SU(N)$ symmetry. We show that, just as in the abelian case, it is possible to bosonize the low-energy regime of the theory. It is also a gauge theory and, as expected, it is non-abelian and it is closely related to a (level $k=1$) $SU(N)$ Chern-Simons gauge theory. Naively, one would have expected that in the non-abelian case the bosonic theory should be the Yang-Mills-Chern-Simons (YMCS) theory. Surprisingly, we find that the bosonized theory is related, but not identical, to YMCS. However, in the limit of weak (fermionic-) coupling $g \to 0$, the bosonic theory becomes equal to the CS gauge theory. At first sight it may seem surprising that a simple theory such as free massive fermions may be equivalent to YMCS. It may be noted that also in $1+1$ dimensions the bosonized theory of a massive free fermion is also non-trivial: it is equivalent to a special case of the sine-gordon theory [2, 3] (in the abelian case) and to a perturbed Wess-Zumino-Witten theory [4].

We will follow the same strategy as in reference [1] and seek a bosonic theory which reproduces correctly the *low-energy* regime of the massive fermionic theory. It should be stressed that this procedure is, in a sense, opposite to what is done in $1+1$ dimensions. There, bosonization is a set of operator identities valid at length scales *short* compared with the Compton wavelength of the fermions. Here, instead, we only consider the long distance regime.

This paper is organized as follows. In Section 2 we present the mapping of the low energy sector of the non-abelian fermionic theory into an equivalent gauge theory. In Section 3 we derive a set of identities for the fermionic currents. In Section 4 we discuss the role of the Wilson loops of the gauge theory in the equivalent Fermi theory. In Section5 we draw a few conclusions on the mapping presnted here.

2 The Mapping

We start from the three-dimensional (Euclidean) massive $SU(N)$ Thirring model Lagrangian:

$$\mathcal{L}_{Th} = \bar{\psi}(i\partial\!\!\!/ + m)\psi - \frac{g^2}{2} j^{a\mu} j^a_\mu \tag{1}$$

where ψ is a two-component Dirac spinor in the fundamental representation of $SU(N)$, and $j^{a\mu}$ the $SU(N)$ current,

$$j^{a\mu} = \bar{\psi}^i t^a_{ij} \gamma^\mu \psi^j. \tag{2}$$

The coupling constant g^2 has dimensions of inverse mass. (Although non-renormalizable by power counting, four fermion interaction models in $2+1$ dimensions are known to be renormalizable in the $1/N$ expansion [5].) We normalize the $SU(N)$ generators according to $\mathrm{tr}\, t^a t^b = \delta^{ab}$.

The partition function for the theory is defined as

$$\mathcal{Z}_{Th} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{-\int d^3x\, [\bar{\psi}(i\partial\!\!\!/ + m)\psi - \frac{g^2}{2} j^{a\mu} j_{a\mu}]\}. \tag{3}$$

We eliminate the quartic interaction by introducing a vector field a_μ taking values in the Lie algebra of $SU(N)$, through the identity

$$\exp\{\frac{g^2}{2}\int d^3x\; j^{a\mu} j^a_\mu\} = \int \mathcal{D}a_\mu \exp\{-\,\mathrm{tr}\int d^3x\, (\frac{1}{2g^2} a^\mu a_\mu + j^\mu a_\mu)\} \tag{4}$$

(up to a multiplicative normalization constant), so that the partition function becomes

$$\mathcal{Z}_{Th} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}a_\mu \exp\{-\int d^3x\, [\bar{\psi}(i\partial\!\!\!/ + m + a\!\!\!/)\psi + \frac{1}{2g^2}\,\mathrm{tr}\, a^\mu a_\mu]\}. \tag{5}$$

The fermionic path-integral can now be done and it yields, as usual, the determinant of the Dirac operator

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{-\int d^3x\, \bar{\psi}(i\partial\!\!\!/ + m + a\!\!\!/)\psi\} = \det(i\partial\!\!\!/ + m + a\!\!\!/) \tag{6}$$

Being the Dirac operator unbounded, its determinant requires regularization. Any sensible regularization approach (for example, ζ-function or Pauli-Villars) gives a parity violating contribution [6]-[8]. There are also parity conserving terms which have been computed as an expansion in inverse powers of the fermion mass:

$$\ln\det(i\partial\!\!\!/ + m + a\!\!\!/) = \pm\frac{1}{16\pi}S_{CS}[a] + I_{PC}[a] + O(\partial^2/m^2), \tag{7}$$

where the Chern-Simons action S_{CS} is given by

$$S_{CS}[a] = \int d^3x\, i\epsilon_{\mu\nu\lambda}\,\mathrm{tr}\int d^3x\,(f_{\mu\nu}a_\lambda - \frac{2}{3}a_\mu a_\nu a_\lambda). \tag{8}$$

Concerning the parity conserving contributions, one has

$$I_{PC}[a] = -\frac{1}{24\pi m}\,\mathrm{tr}\int d^3x\, f^{\mu\nu}f_{\mu\nu} + \cdots, \tag{9}$$

where

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu]. \tag{10}$$

Using this result we can write Z_{Th} in the form

$$Z_{Th} = \int \mathcal{D}a_\mu \exp(-S_{eff}[a]), \tag{11}$$

where S_{eff} is given by

$$S_{eff}[a] = \frac{1}{2g^2}\,\mathrm{tr}\int d^3x\, a_\mu a^\mu \mp \frac{1}{16\pi}S_{CS}[a] + \frac{1}{24\pi m}\,\mathrm{tr}\int d^3x\, f^{\mu\nu}f_{\mu\nu} + O(\partial^2/m^2). \tag{12}$$

Up to corrections of order $1/m$, we recognize in S_{eff} the non-abelian version of the self-dual action S_{SD} introduced some time ago by Townsend, Pilch and van Nieuwenhuizen [9],

$$S_{SD}[a] = \int d^3x\,\frac{1}{2g^2}\,\mathrm{tr}\, a_\mu a^\mu \mp \frac{1}{16\pi}S_{CS}[a]. \tag{13}$$

Then, to leading order in $1/m$ we have established the following identification:

$$Z_{Th} \approx Z_{SD} = \int \mathcal{D}a_\mu \exp(-S_{SD}[a]) \tag{14}$$

In the abelian case, it has been shown by Deser and Jackiw [10] that the model with dynamics defined by S_{SD} is equivalent to the Maxwell-Chern-Simons (MCS) theory. Using this connection, we have shown in [1] the equivalence of the abelian Thirring model and the MCS theory. Our proof was based in the use of an "interpolating Action" S_I connecting the self dual and the MCS actions. It has been already recognized in [10] that the non-abelian extension of these kind of equivalences is more involved. In fact, we shall show here that the non-abelian self-dual action (and, consequently, the $SU(N)$ Thirring model) is not equivalent to a Yang-Mills-Chern-Simons theory (the natural extension of the abelian MCS theory) but to a model where, instead of the Yang-Mills action, one has a more complicated term, which reduces to the quadratic $\operatorname{tr} F_{\mu\nu}^2$ only in the $g^2 \to 0$ limit.

To see this, we consider, following [13], the interpolating Lagrangian

$$L_I[a, A] = \frac{1}{8g^2} \operatorname{tr} a_\mu^2 \pm \frac{i}{16\pi} \epsilon_{\mu\nu\lambda} \operatorname{tr} a_\mu (F_{\nu\lambda} + A_\nu a_\lambda) \pm \frac{1}{16\pi} L_{CS}[A], \qquad (15)$$

with $F_{\mu\nu}$ the non-Abelian curvature

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \qquad (16)$$

and $L_{CS}[A]$ is the Chern-Simons Lagrangian. The partition function associated to L_I is defined as

$$Z_I = \int \mathcal{D}A_\mu \mathcal{D}a_\mu \exp\{-\int d^3x\, L_I[a, A]\}. \qquad (17)$$

In order to see the connection between Z_I and the partition function Z_{SD} for the self-dual system, we shift in (17) the A_μ variables as follows:

$$A_\mu = \bar{A}_\mu - a_\mu. \qquad (18)$$

Under this change, one has

$$F_{\mu\nu}[A] = F_{\mu\nu}[\bar{A}] - D[\bar{A}]_{[\mu} a_{\nu]} + [a_\mu, a_\nu], \qquad (19)$$

with the covariant derivative defined as

$$D_\mu[\bar{A}] = \partial_\mu + [\bar{A}_\mu,\ ]. \qquad (20)$$

Concerning the Chern-Simons Lagrangian, one has

$$L_{CS}[A] = L_{CS}[\bar{A}] - 2\epsilon_{\mu\nu\lambda} a_\mu (F_{\nu\lambda}[\bar{A}] - D_\nu[\bar{A}] a_\lambda + \frac{2}{3} a_\nu a_\lambda). \qquad (21)$$

Puting all this together we easily find

$$L_I[a,A] = \frac{1}{2g^2}\,\mathrm{tr}\,a_\mu a^\mu \pm \frac{1}{16\pi}L_{CS}[\bar{A}] \mp i\frac{1}{8\pi}\,\mathrm{tr}\,\epsilon_{\mu\nu\lambda}\,a_\mu(\partial_\nu a_\lambda + \frac{2}{3}a_\nu a_\lambda). \quad (22)$$

Transformation (18) has unit Jacobian and completely decouples the $\bar{A}$ field. Denoting by N the result of integrating over $\bar{A}$, one then has

$$\begin{aligned} Z_I &= N\int \mathcal{D}a_\mu \exp\{-\frac{1}{2g^2}\,\mathrm{tr}\int d^3x\, a_\mu a^\mu\} \times \\ &\quad \exp\{\pm i\frac{1}{8\pi}\,\mathrm{tr}\int d^3x\,\epsilon_{\mu\nu\lambda}a_\mu(\partial_\nu a_\lambda + \frac{2}{3}a_\nu a_\lambda)\}. \end{aligned} \quad (23)$$

We recognize in this expression the action for the self-dual system defined in eqs. (13). Then, we have proven that

$$Z_I = N\int \mathcal{D}a_\mu \exp(-S_{SD}). \quad (24)$$

Comparing eqs. (14) and (24) we can establish the following relation

$$Z_{Th} \approx Z_I. \quad (25)$$

We shall now proceed to perform the path-integrations in Z_I in the inverse order, that is, first integrating over a_μ. To this end, starting again from eq. (15) we write

$$Z_I = \int \mathcal{D}A_\mu \exp(\mp\frac{1}{16\pi}S_{CS}[A]) \times \exp(-I[A]), \quad (26)$$

with

$$\begin{aligned} \exp(-I[A]) &= \int \mathcal{D}a_\mu \exp\{-\frac{1}{8g^2}\,\mathrm{tr}\int d^3x\, a_\mu a^\mu\} \times \\ &\quad \exp\{\mp i\frac{1}{16\pi}\,\mathrm{tr}\int d^3x\,\epsilon_{\mu\nu\lambda}a_\mu(F_{\nu\lambda} + A_\nu a_\lambda)\}. \end{aligned} \quad (27)$$

We can rewrite $I[A]$ in the form

$$\exp(-I[A]) = \int \mathcal{D}a_\mu \exp\{-\,\mathrm{tr}\int d^3x\,(\frac{1}{2}a_\mu S^{-1}_{\mu\nu}a_\nu \pm \frac{i}{8\pi}{}^*F_\mu a_\mu)\}, \quad (28)$$

with

$$S^{-1ab}_{\mu\nu} = \frac{1}{4g^2}\delta_{\mu\nu}\delta^{ab} \mp \frac{i}{8\pi}\epsilon_{\mu\nu\lambda}f^{acb}A^c_\lambda \tag{29}$$

and

$${}^*F_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda}F_{\nu\lambda}. \tag{30}$$

Then, if we perform the path-integral over a_μ we find

$$I[A] = \frac{1}{128\pi^2}\,\mathrm{tr}\int d^3x\,{}^*F_\mu S_{\mu\nu}{}^*F_\nu. \tag{31}$$

Hence, we see that Z_I can be written in the form

$$Z_I = \int \mathcal{D}A_\mu \exp(-S_{FCS}[A]) \equiv Z_{FCS} \tag{32}$$

where

$$S_{FCS}[A] = \pm\frac{1}{16\pi}S_{CS}[A] + \frac{1}{128\pi^2}\,\mathrm{tr}\int d^3x\,{}^*F_\mu S_{\mu\nu}{}^*F_\nu \tag{33}$$

Then, using eq. (25) we have the relation

$$Z_{Th} \approx Z_{FCS} \tag{34}$$

which shows the equivalence (to order $1/m$) of the fermionic Thirring model and a bosonic theory with action S_{FCS}. Let us notice that only in the limit $g^2 \to 0$ the action S_{FCS} reduces to the Yang-Mills-Chern-Simons action.

3 Bosonization Identities

In order to infer the bosonization identities which derive from the equivalence found in the last section, we add a source for the Thirring current:

$$L_{Th}[b_\mu] = L_{Th} + \mathrm{tr}\int d^3x\, j^\mu b_\mu \tag{35}$$

Then, instead of (5), the partition function now reads

$$\mathcal{Z}_{Th}[b] = \int \mathcal{D}\bar\psi\mathcal{D}\psi\mathcal{D}a_\mu \exp\{-\int d^3x\,[\bar\psi(i\partial\!\!\!/ + m + a\!\!\!/ + b\!\!\!/)\psi + \frac{1}{2g^2}\,\mathrm{tr}\,a^\mu a_\mu]\}. \tag{36}$$

Or, after shifting $a_\mu \to a_\mu - b_\mu$,

$$Z_{Th}[b] = \exp\{-\frac{1}{2g^2}\,\mathrm{tr}\int d^3x\, b_\mu b^\mu\} \times \int \mathcal{D}a_\mu \exp\{-S_{eff}[a] + \frac{1}{g^2}\,\mathrm{tr}\int d^3x\, b_\mu a^\mu\} \tag{37}$$

with $S_{eff}[a]$ still given by eq. (12). Then, we can again establish to order $1/m$ the connection between the Thirring and self-dual models, in the presence of sources:

$$Z_{Th}[b] = \exp\{-\frac{1}{2g^2}\,\mathrm{tr}\int d^3x\, b_\mu b^\mu\} \times \int \mathcal{D}a_\mu \exp\{-S_{SD}[a] + \frac{1}{g^2}\,\mathrm{tr}\int d^3x\, b_\mu a^\mu\}. \tag{38}$$

or

$$Z_{Th}[b] = \exp\{-\frac{1}{2g^2}\,\mathrm{tr}\int d^3x\, b_\mu b^\mu\} \times Z_{SD}[b], \tag{39}$$

where

$$Z_{SD}[b] = \int \mathcal{D}a_\mu \exp\{-S_{SD}[a] + \frac{1}{g^2}\,\mathrm{tr}\int d^3x\, b_\mu a^\mu\} \tag{40}$$

In order to connect this with the vector boson system, let us again consider the interpolating Lagrangian L_I (eq. 15), but now in the presence of sources:

$$L_I[a, A, b] = L_I[a, A] + \frac{1}{4g^2}\,\mathrm{tr}\, a_\mu b^\mu. \tag{41}$$

By shifting as before $A_\mu \to A_\mu - a_\mu$ and integrating out A_μ, one easily shows that the corresponding partition function $Z_I[b]$ coincides (up to a normalization factor) with $Z_{SD}[b]$,

$$Z_{SD}[b] = Z_I[b] \tag{42}$$

and then

$$Z_{Th}[b] = \exp\{-\frac{1}{2g^2}\,\mathrm{tr}\int d^3x\, b^\mu b_\mu\} \times Z_I[b]. \tag{43}$$

If we integrate in the inverse order we have (after shifting $a_\mu \to 2a_\mu$),

$$Z_I[b] = \int \mathcal{D}A_\mu \exp(\mp\frac{1}{16\pi} S_{CS}[A]) \times \exp(-I[A, b]) \tag{44}$$

with

$$\exp(-I[A, b]) = \int \mathcal{D}a_\mu \exp\{-\,\mathrm{tr}\int d^3x\, [\frac{1}{2} a_\mu S^{-1}_{\mu\nu} a_\nu \pm \frac{i}{8\pi}({}^*F_\mu \mp i\frac{8\pi}{g^2} b_\mu) a_\mu]\}. \tag{45}$$

This means that $Z_I[b]$ can be written in the form

$$
\begin{aligned}
Z_I[b] &= \int \mathcal{D}A_\mu \exp(\mp\frac{1}{16\pi} S_{CS}[A]) \times \\
&\quad \exp\{-\frac{1}{128\pi^2}\,\mathrm{tr}\int d^3x\,({}^*F_\mu \mp i\frac{8\pi}{g^2}b_\mu)S_{\mu\nu}({}^*F_\nu \mp i\frac{8\pi}{g^2}b_\nu)\}, \quad (46)
\end{aligned}
$$

so we have the relation

$$
\begin{aligned}
Z_{Th}[b] &= \exp\{-\frac{1}{2g^2}\,\mathrm{tr}\int d^3x\, b^\mu b_\mu\} \times \int \mathcal{D}A_\mu \exp(-S_{FCS}[A]) \times \quad (47) \\
&\quad \exp\{\frac{1}{8g^4}\,\mathrm{tr}\int d^3x\, b^\mu S_{\mu\nu} b^\nu\} \times \exp\{\pm\frac{i}{16\pi g^2}\,\mathrm{tr}\int d^3x\, b^\mu S_{\mu\nu}{}^*F^\nu\}.
\end{aligned}
$$

This is, in its most general form, the result we were after. It provides a complete low-energy bosonization prescription, valid for any g^2, of the matrix elements of the fermionic current. Since after differentiating we must set $b_\mu = 0$, we see that, as suggested by eq. (34), the bosonized version of the Thirring model is described by the action S_{FCS} in eq. (33). However, the bosonization rule for the fermionic current is not simple. For instance, from eq. (47) we get, up to contact terms,

$$
j^a_\mu \to \pm\frac{i}{16\pi g^2} S^{ab}_{\mu\nu}{}^*F^\nu_{\ b}, \quad (48)
$$

However, higher derivatives respect to the sources lead to more involved bosonic equivalents of the vacuum expectation value of products of fermionic currents.

This more complex structure should not come as a surprise: even in two-dimensions a simple bosonization procedure applies only to free fermions [4]. Then, with that in mind, we restrict the discussion that follows to the $g^2 \to 0$ limit, where with eq. (29), eqs. (33, 47) reduce to

$$
\begin{aligned}
Z_{Th}[b] &= \int \mathcal{D}A_\mu \exp(\mp\frac{1}{16\pi} S_{CS}[A]) \times \exp\{\pm\frac{i}{4\pi}\,\mathrm{tr}\int d^3x\, b_\mu{}^*F^\mu\} \times \\
&\quad \exp\{\pm\frac{i}{4\pi}\,\mathrm{tr}\int d^3x\, \epsilon^{\mu\alpha\nu} b_\mu [A_\alpha, b_\nu]\}, \quad (49)
\end{aligned}
$$

so, in that limit, the ground state fermionic current maps to

$$
j^a_\mu \to \pm\frac{i}{4\pi}{}^*F^a_{\ \mu}, \quad (50)
$$

Concerning the factor in (49) which is quadratic in the sources, its first contribution will arise when computing current-current correlation functions. One can see however that the resulting commutator algebra, obtained via the Bjorken-Johnson-Low method from these correlation functions, is not modified by the quadratic term. In this sense, eq.(50) gives the bosonization mapping for non-abelian free fermions in 3 dimensions as the natural generalization of the result obtained in ref.[1] for the abelian case. Of course the fact that our results only hold at long distances makes the analysis of the commutator algebra, which tests short distances, not completely reliable.

We thus see that the non-abelian bosonization of free $SU(N)$ massive fermions in $2+1$ dimensions leads to the (level $k=1$) $SU(N)$ Chern-Simons theory, with the fermionic current being mapped to the dual of the gauge field strength. As stated earlier, this result holds only for length scales large compared with the Compton wavelength of the fermion, since our results were obtained for large fermion mass. It is important to notice that the limit $g^2 \to 0$ to which we restrict henceforth, corresponds to free fermions but not to an abelian gauge theory. On the contrary. $F^a_{\mu\nu}$ is the full non-abelian field strength (cf. eq. (16)) and the Yang-Mills coupling is proportional to $1/g^2 \to \infty$, which is why we are left with a pure Chern-Simons theory and not with a mixed Yang-Mills-Chern-Simons action.

Eq. (50) gives a natural non-abelian extension of the abelian bosonization rule obtained in ref. [1]. In the abelian case one can interpret the $(2+1)$-dimensional bosonization formula (which is identical to eq. (50) but with $F_{\mu\nu}$ the abelian curvature) as the analog of the $(1+1)$-dimensional result $\bar\psi\gamma^\mu\psi \to (1/\sqrt{\pi})\epsilon_{\mu\nu}\partial^\nu\phi$. (The factor of i in the expression for the current in eq. (50) appears because we are working in Euclidean space). To establish this correspondence also in the non-abelian model, one should remember that in this last case the two dimensional bosonization identity reads [4]

$$j_+ \to -\frac{i}{4\pi} h^{-1}\partial_+ h \tag{51}$$

$$j_- \to -\frac{i}{4\pi} h\partial_- h^{-1} \tag{52}$$

with fermions in the fundamental representation of the group G, and h an element of G.

To make contact with our $(2+1)$-dimensional result, let us first note that in the $g^2 \to 0$ limit the resulting bosonic action (the Chern-Simons action) is gauge invariant and so a gauge fixing is required. The natural gauge in order to compare the results in $1+1$ and $2+1$ dimensions is the $A_3 = 0$ gauge. Moreover, one can write

$$A_\pm = A_1 \pm i A_2 \tag{53}$$

in the form

$$A_+ = -i h^{-1} \partial_+ h \tag{54}$$

$$A_- = -i f^{-1} \partial_- f \tag{55}$$

With this, one has from (50),

$$j_+ \to \pm \frac{i}{4\pi} \partial_3 (h^{-1} \partial_+ h) \tag{56}$$

$$j_- \to \mp \frac{i}{4\pi} \partial_3 (f^{-1} \partial_- f) \tag{57}$$

These are the $(2+1)$-dimensional analogs of the two-dimensional formulas (51)-(52). Concerning the additional component j_3, one has

$$j_3 = \pm \frac{i}{8\pi} \Big(\partial_+ (f^{-1} \partial_- f) - \partial_- (h^{-1} \partial_+ h) + i (h^{-1} \partial_+ h, f^{-1} \partial_- f) \Big) \tag{58}$$

4 Wilson Loops

Perhaps the most interesting aspect of the bosonization identities of eqs. (49)-(50) is the promotion of the global $SU(N)$ symmetry of the free fermions to a local gauge symmetry in the bosonic theory. To explore the contents of this bosonization rule we consider the natural objects in the gauge theory, namely the vacuum expectation value of Wilson loops. In the Chern-Simons theory they measure topological invariants determined by the linkings of the loops and by the topology of the base manifold [15]. For one loop Γ,

$$W[\Gamma] = \mathrm{tr}\, P \exp(i \oint_\Gamma dx^\mu \, A_\mu) \tag{59}$$

where P denotes the path ordering of the exponential, and the trace is taken in the representation carried by the loop. According to the bosonization prescription of eq. (50), to relate this operator to the fermionic theory we must express $W[\Gamma]$ in terms of the field strength $F_{\mu\nu}$ rather than the potential

A_μ. In the abelian case this can always be done by means of Stokes theorem. As discussed in ref. [1], this leads to an explicit mapping between abelian Wilson loops and non-local fermionic operators. Hence, in this way, the latter are related to the linking of loops and thus probe the generalized statistics of the external particles that propagate along those loops. One way to extend that analysis to the non-abelian case is to use the non-abelian extension of Stokes theorem developed in [16]. For an arbitrary loop $\Gamma = \partial\Sigma$, the boundary of a surface Σ, one has

$$W[\partial\Sigma] = \mathrm{tr}\, P_t \exp\{i\int_0^1 dt \int_0^1 ds \frac{\partial\Sigma^\mu}{\partial s}\frac{\partial\Sigma^\nu}{\partial t} W^{-1}[{}_s\Sigma(t)_0] F_{\mu\nu}(\Sigma(t,s)) W[{}_s\Sigma(t)_0]\} \tag{60}$$

Here Σ is looked upon as a sheet, that is, a one parameter family of paths parametrized by t, $0 \le t \le 1$. For each t, $\Sigma(t)$ is a path, itself parametrized by s, $0 \le s \le 1$, with fixed end-points: $\partial\Sigma(t,s)/\partial t = 0$ at $s = 0, 1$. For a given t, ${}_s\Sigma(t)_0$ denotes the segment of the path $\Sigma(t)$ connecting the points $\Sigma(t,0)$ and $\Sigma(t,s)$, and $W[{}_s\Sigma(t)_0]$ is the corresponding (open) Wilson line. Finally, P_t in eq. (60) denotes ordering of the t integral, while the s integral is not ordered (although there is an s-ordering inside each $W[{}_s\Sigma(t)_0]$.)

In the abelian case the two open Wilson lines $W[{}_s\Sigma(t)_0]$ in eq. (60) cancel each other and one recovers the usual Stokes theorem, involving only the gauge field strength. In the non-abelian case, however, the factors $W[{}_s\Sigma(t)_0]$ are needed for gauge invariance, and introduce an explicit dependence of the Wilson loop operator on the gauge potential A_μ. Thus, as opposed to the abelian case, the non-abelian Wilson loop operator cannot be mapped in a straightforward way to a fermionic operator through the bosonization rule in eq. (50).

For planar loops this difficulty is only apparent. Indeed, consider $W[\partial\Sigma]$, with Σ contained, say, in the $(1,2)$ plane. Imposing the $A_3 = 0$ gauge condition, there is a remnant gauge freedom for the A_1 and A_2 components in the $(1,2)$ plane, which is the symmetry of a 2-dimensional gauge theory in that plane. As discussed in [16], one can use that gauge symmetry, together with the freedom of parametrization of the surface Σ, so the open Wilson line elements in the right hand side of eq. (60) become the identity. More precisely, choosing the gauge condition $A_2 = 0$ on the Σ-plane, eq. (60) can be simplified to

$$W[\partial\Sigma] = \mathrm{tr}\, P_t \exp\{i\int_0^1 dt \int_0^1 ds \frac{\partial\Sigma^\mu}{\partial s}\frac{\partial\Sigma^\nu}{\partial t} F_{\mu\nu}(\Sigma(t,s))\} \tag{61}$$

provided that Σ is parametrized so as to have $\partial\Sigma/\partial t$ and $\partial\Sigma/\partial s$ parallel to the x_1 and x_2 axis, respectively. This apparent breaking of rotational invariance, which includes the presence of t-ordering but not of s-ordering, is a consequence of the $A_2 = 0$ gauge condition on the Σ-plane, and will be removed by the functional integral over the gauge fields. Then, writing

$$\mathcal{J}[\Sigma] = \mathrm{tr}\, P_t \exp\{\pm 4\pi \int_0^1 dt \int_0^1 ds \frac{\partial\Sigma^\mu}{\partial s}\frac{\partial\Sigma^\nu}{\partial t} \epsilon_{\mu\nu\lambda} j_\lambda[\Sigma(t,s)]\} \tag{62}$$

with j_μ the fermionic current in eq. (2), the bosonization formula (50) gives

$$\langle \mathcal{J}[\Sigma]\rangle_{ferm} = \langle W[\partial\Sigma]\rangle_{CS} \tag{63}$$

where in the left hand side the subindex 'ferm' stands for free fermions. This is the non-abelian generalization of the result obtained in [1]. It relates a suitably defined non-abelian flux of the fermionic current through a flat surface, and the Wilson loop associated to the boundary of that surface, with both quantities in the same representation of the group.

It should be stressed that the bosonic side of this relation is, by definition, independent of the surface Σ and its parametrization. In the fermionic side, however, this is not obvious. The relation was derived assuming a flat surface Σ, and it is tempting to assume that this may be extended to smooth deformations away from the plane. But more relevant is the apparent breaking of rotational invariance in the fermionic side due to the remaining t-ordering in eq. (62). This should certainly be expected to be taken care of by the particular parametrization assumed above for Σ. Indeed, one should expect that the very need of a parametrization and of a matching ordering of the surface integral of the fermionic current, is just a limitation of our present analysis. In addition, as is well known, the expectation value of the Wilson loop is singular and must be regularized. A natural and consistent regularization scheme is provided by the framing of the loop [15]. In the case of a non-intersecting loop on a plane, considered here, that framing can be chosen also as a plane loop, not intersecting itself nor the original loop. Again, it is not clear at this point how these singularities in the bosonic side will show up in the (free) fermionic side, and what role will the framing play from the fermionic point of view.

It is natural to ask whether this analysis can be extended to several loops and their possible linkings, as done in [1] for the Abelian case. In the bosonic side one is interested in the expectation value $\langle W[\Gamma_1]W[\Gamma_2]\rangle_{CS}$ or, better yet, the ratio

$$\frac{\langle W[\Gamma_1]W[\Gamma_2]\rangle_{CS}}{\langle W[\Gamma_1]\rangle_{CS}\langle W[\Gamma_2]\rangle_{CS}} \tag{64}$$

For non intersecting loops this is a well defined, non singular object in the Chern-Simons theory, which depends only on the linking of the two loops Γ_1 and Γ_2 [15]. Assuming this to be non-trivial (and non-singular), the two loops cannot be flat and lying on the same plane, so the previous construction fails. But once the ratio (64) has been computed in the Chern-Simons theory, we can take the limit in which the two loops collapse onto a single plane. This is a singular limit in which the loops necessarily intersect each other. Their linking is not well defined any more, and the value of (64) depends on the initial non-singular loops used in the computation. However, at the classical level, before the functional integral is performed, we can repeat the previous construction with no difficulties for any arrangement of loops on the plane [16]. Thus, formally we can write

$$\frac{\langle \mathcal{J}[\Sigma_1\cup\Sigma_2]\rangle_{ferm}}{\langle \mathcal{J}[\Sigma_1]\rangle_{ferm}\langle \mathcal{J}[\Sigma_2]\rangle_{ferm}} = \frac{\langle W[\partial\Sigma_1]W[\partial\Sigma_2]\rangle_{CS}}{\langle W[\partial\Sigma_1]\rangle_{CS}\langle W[\partial\Sigma_2]\rangle_{CS}} \tag{65}$$

where both surfaces Σ_1 and Σ_2 are contained in the same plane. As we just stated, the bosonic side of this relation will be ill defined in general. But it can be given a well defined meaning by lifting the loops $\partial\Sigma_i$ from the plane to non intersecting three-dimensional loops Γ_i. This can be done in different ways, specifying different linkings of the loops Γ_i compatible with the intersections of their projections $\partial\Sigma_i$ onto the plane. Correspondingly, in the fermionic side, the surface $\Sigma_1\cup\Sigma_2$ must be complemented with a prescription stating the way in which the two surfaces Σ_i overlap. The different possible liftings of the loops specify different overlaps of the surfaces, as ilustrated in Fig. (1). In this way, relation (65) (and its generalizations) can be viewed as a defining relation, through bosonization, of the vaccuum expectation value of the flux of the fermionic current through surfaces with foldings.

5 Conclusions

We close with a few remarks on the nature of the mapping discussed here. In this paper we showed that the low energy sector of the massive $SU(N)$ Thirring model in 2+1 dimensions is equivalent to the long distance regime of a non-abelian gauge theory, closely related to the Yang-Mills-Chern-Simons gauge theory. In the weak coupling limit, the two theories become identical. It is worthwile to stress that this mapping only holds at long distances. In that regime, the gauge theory is a topological field theory and so is the fermion theory. Secondly, just as in the abelian case, we discover the existence of operators of the Fermi theory which ought to exhibit fractional statistics. However, unlike the abelain theory, these objects are substantially more complex. Finally, the bosonic theory is, essentially, a level $k = 1$ $SU(N)$ Chern-Simons theory. It would be interesting to find a fermionic analog of a Chern-Simnos theory with level higher than one.

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Figure 1: Different overlaps of the surfaces Σ_1 and Σ_2 on a plane, determined by the possible liftings of the loops $\partial\Sigma_1$ and $\partial\Sigma_2$ away from the plane.